

Backward Stochastic Differential Equations with Continuous Coefficients in a Markov Chain Model and with Applications to European Options

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Abstract

In this paper we discuss backward stochastic differential equations with Markov chain noise, having continuous drivers. We obtain the existence of a solution which is possibly not unique. Moreover, we show there is a minimal solution for this kind of equation and derive the corresponding comparison result. This is applied to pricing of European options in a market with Markov chain noise.

1 Introduction

Backward stochastic differential equations (BSDEs) have been used as pricing and hedging tools in Finance. Applications of BSDEs in Finance are usually focused on a market where prices follow geometric Brownian motion or other related diffusion process models. Hence the BSDEs in such cases are driven by Brownian motions. We particularly mention the works of El Karoui and Quenez [5] and [6].

The first work of Pardoux and Peng [12] on general BSDEs, considers equations of the form:

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s, \quad t \in [0, T],$$

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where B is a Brownian Motion, g is the driver, or drift coefficient, and $g(t, y, z)$ is Lipschitz continuous in the variables y and z . In this case, the solution of the BSDE is unique. In derivative pricing and hedging, this leads to a unique hedging strategy and a unique price. In some other market models, one needs to deal with BSDEs with non-Lipschitz drivers. Lepeltier and San Martin [11] discussed existence of solution of such BSDEs and showed the existence of a minimal solution.

All of the above references discuss BSDEs driven by Brownian motion or related jump-diffusion process. However, it is known from the work of Kushner [10] that such processes can be approximated by Markov chain models. Hence, there is a motivation for discussing Markov chain model. van der Hoek and Elliott [15] introduced a market model where uncertainties are modeled by a finite state Markov chain, rather than Brownian motion or related jump diffusions. The Markov chain has a semimartingale representation involving a vector martingale $M = \{M_t \in \mathbb{R}^N, t \geq 0\}$. BSDEs in this framework were introduced by Cohen and Elliott [2] as

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z'_s dM_s, \quad t \in [0, T],$$

where f is Lipschitz in y and z . We derived a new comparison theorem in Yang, Ramarimbahoaka and Elliott [17] which we think is easier to use in this framework than the Comparison results found in Cohen and Elliott [3] which consider a more general case.

In this paper, using the comparison theorem from Yang, Ramarimbahoaka and Elliott [17], we discuss BSDEs in the Markov chain framework with a continuous driver f which has a linear growth in y and Lipschitz in z . We follow the method in Lepeltier and San Martin [11], that is we construct a monotone sequence of Lipschitz continuous functions of y and z such that they converge to f . The existence of solutions will be established followed by the existence of a minimal solution and a corresponding comparison result. An application is given to European option pricing in a market where randomness is modelled by a Markov chain and consumption of investors is taken into account.

The present paper is structured as follows: Section 2 will present the model and some preliminary results. Section 3 discusses the existence of multiple solutions of BSDEs with a continuous driver, as well as the minimal solution, followed by the corresponding comparison result. The final section consists of an application to European options.

2 The Model and Some Preliminary Results

Let $T > 0$ and $N \in \mathbb{N}$ be two constants. Consider a finite state Markov chain. Following van der Hoek and Elliott [15], we assume the finite state Markov chain $X = \{X_t, t \geq 0\}$ is defined on the probability space (Ω, \mathcal{F}, P) and the state space of X is identified with the set of unit vectors $\{e_1, e_2, \dots, e_N\}$ in \mathbb{R}^N , where $e_i = (0, \dots, 1, \dots, 0)'$ with 1 in the i -th position. Then the Markov chain has the semimartingale representation:

$$X_t = X_0 + \int_0^t A_s X_s ds + M_t. \quad (1)$$

Here, $A = \{A_t, t \geq 0\}$ is the rate matrix of the chain X and M is a vector martingale (See Elliott, Aggoun and Moore [8]). We assume the elements $A_{ij}(t)$ of $A = \{A_t, t \geq 0\}$ are bounded for all $t \in [0, T]$. Then the martingale M is square integrable.

Take $\mathcal{F}_t = \sigma\{X_s; 0 \leq s \leq t\}$ to be the σ -algebra generated by the Markov process $X = \{X_t\}$ and $\{\mathcal{F}_t\}$ to be its filtration. Since X is right continuous and has left limits, (written by RCLL), the filtration $\{\mathcal{F}_t\}$ is also right-continuous.

The following is the product rule for semimartingales and we refer the reader to [7] for proof:

Lemma 2.1 (Product Rule for Semimartingales). *Let Y and Z be two scalar RCLL semimartingales, with no continuous martingale part. Then*

$$Y_t Z_t = Y_T Z_T - \int_t^T Y_{s-} dZ_s - \int_t^T Z_{s-} dY_s - \sum_{t < s \leq T} \Delta Z_s \Delta Y_s.$$

Here, $\sum_{0 < s \leq t} \Delta Z_s \Delta Y_s$ is the optional covariation of Y_t and Z_t and is also written as $[Z, Y]_t$.

For our (vector) Markov chain $X_t \in \{e_1, \dots, e_N\}$, note that $X_t X_t' = \text{diag}(X_t)$. Also, $dX_t = A_t X_t dt + dM_t$. By Lemma 2.1, we know for $t \in [0, T]$,

$$\begin{aligned} X_t X_t' &= X_0 X_0' + \int_0^t X_{s-} dX_s' + \int_0^t (dX_s) X_{s-}' + \sum_{0 < s \leq t} \Delta X_s \Delta X_s' \\ &= \text{diag}(X_0) + \int_0^t X_s (A_s X_s)' ds + \int_0^t X_{s-} dM_s' + \int_0^t A_s X_s X_{s-}' ds \\ &\quad + \int_0^t (dM_s) X_{s-}' + [X, X]_t \end{aligned}$$

$$\begin{aligned}
&= \text{diag}(X_0) + \int_0^t X_s X'_s A'_s ds + \int_0^t X_{s-} dM'_s + \int_0^t A_s X_s X'_{s-} ds \\
&+ \int_0^t (dM_s) X'_{s-} + [X, X]_t - \langle X, X \rangle_t + \langle X, X \rangle_t. \tag{2}
\end{aligned}$$

Recall, $\langle X, X \rangle$ is the unique predictable $N \times N$ matrix process such that $[X, X] - \langle X, X \rangle$ is a matrix valued martingale and write

$$L_t = [X, X]_t - \langle X, X \rangle_t, \quad t \in [0, T]. \tag{3}$$

However,

$$X_t X'_t = \text{diag}(X_t) = \text{diag}(X_0) + \int_0^t \text{diag}(A_s X_s) ds + \int_0^t \text{diag}(M_s). \tag{4}$$

Equating the predictable terms in (2) and (4), we have

$$\langle X, X \rangle_t = \int_0^t \text{diag}(A_s X_s) ds - \int_0^t \text{diag}(X_s) A'_s ds - \int_0^t A_s \text{diag}(X_s) ds. \tag{5}$$

Let Ψ be the matrix

$$\Psi_t = \text{diag}(A_t X_t) - \text{diag}(X_t) A'_t - A_t \text{diag}(X_t). \tag{6}$$

Then $d\langle X, X \rangle_t = \Psi_t dt$. For any $t > 0$, Cohen and Elliott [2, 4], define the semi-norm $\|\cdot\|_{X_t}$, for $C, D \in \mathbb{R}^{N \times K}$ as :

$$\begin{aligned}
\langle C, D \rangle_{X_t} &= \text{Tr}(C' \Psi_t D), \\
\|C\|_{X_t}^2 &= \langle C, C \rangle_{X_t}.
\end{aligned}$$

We only consider the case where $C \in \mathbb{R}^N$, hence we introduce the semi-norm $\|\cdot\|_{X_t}$ as:

$$\begin{aligned}
\langle C, D \rangle_{X_t} &= C' \Psi_t D, \\
\|C\|_{X_t}^2 &= \langle C, C \rangle_{X_t}. \tag{7}
\end{aligned}$$

It follows from Equation (5) that

$$\int_t^T \|C\|_{X_s}^2 ds = \int_t^T C' d\langle X, X \rangle_s C.$$

For $n \in \mathbb{N}$, denote by $|\cdot|_n$ the Euclidian norm in \mathbb{R}^n and by $\|\cdot\|_{n \times n}$ the norm in $\mathbb{R}^{n \times n}$ such that $\|\Psi\|_{n \times n} = \sqrt{\text{Tr}(\Psi' \Psi)}$ for any $\Psi \in \mathbb{R}^{n \times n}$.

Lemma 2.2 is Lemma 3.5 in Yang, Ramarimbahoaka and Elliott [17].

Lemma 2.2. For any $C \in \mathbb{R}^N$,

$$\|C\|_{X_t} \leq \sqrt{3m}|C|_N, \quad \text{for any } t \in [0, T],$$

where $m > 0$ is the bound of $\|A_t\|_{N \times N}$, for any $t \in [0, T]$.

Denote by \mathcal{P} , the σ -field generated by the predictable processes defined on (Ω, P, \mathcal{F}) and with respect to the filtration $\{\mathcal{F}_t\}_{t \in [0, \infty)}$. For $t \in [0, \infty)$, consider the following spaces:

$L^2(\mathcal{F}_t) := \{\xi; \xi \text{ is a } \mathbb{R}\text{-valued } \mathcal{F}_t\text{-measurable random variable such that } E[|\xi|^2] < \infty\};$

$L^2_{\mathcal{F}}(0, t; \mathbb{R}) := \{\phi : [0, t] \times \Omega \rightarrow \mathbb{R}; \phi \text{ is an adapted and RCLL process with } E[\int_0^t |\phi(s)|^2 ds] < +\infty\};$

$P^2_{\mathcal{F}}(0, t; \mathbb{R}^N) := \{\phi : [0, t] \times \Omega \rightarrow \mathbb{R}^N; \phi \text{ is a predictable process with } E[\int_0^t \|\phi(s)\|_{X_s}^2 ds] < +\infty\}.$

Lemma 2.3 can be found in Ramarimbahoaka, Yang and Elliott [13].

Lemma 2.3. For $t \in [0, T]$ and $Z \in P^2_{\mathcal{F}}(0, t; \mathbb{R}^N)$, the following equation holds:

$$E[(\int_0^t Z'_s dM_s)^2] = E[\int_0^t \|Z_s\|_{X_s}^2 ds].$$

Lemma 2.4 (Theorem 6.2 in Cohen and Elliott [2]) gives the existence and uniqueness result of solutions to BSDEs driven by Markov chains.

Lemma 2.4. Consider the BSDE with Markov chain noise as follows:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z'_s dM_s, \quad t \in [0, T]. \quad (8)$$

Assume $\xi \in L^2(\mathcal{F}_T)$ and the predictable function $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ satisfies a Lipschitz condition, in the sense that: there exists two constants $l_1, l_2 > 0$ such that for each $y_1, y_2 \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{R}^N$, $t \in [0, T]$,

$$|f(t, y_1, z_1) - f(t, y_2, z_2)| \leq l_1 |y_1 - y_2| + l_2 \|z_1 - z_2\|_{X_t}. \quad (9)$$

We also assume f satisfies

$$E[\int_0^T |f^2(t, 0, 0)| dt] < +\infty. \quad (10)$$

Then there exists a solution $(Y, Z) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}) \times P^2_{\mathcal{F}}(0, T; \mathbb{R}^N)$ to BSDE (8). Moreover, this solution is unique up to indistinguishability for Y and equality $d\langle X, X \rangle_t \times \mathbb{P}$ -a.s. for Z .

See Campbell and Meyer [1] for the following definition:

Definition 2.5 (Moore-Penrose pseudoinverse). *The Moore-Penrose pseudoinverse of a square matrix Q is the matrix Q^\dagger satisfying the properties:*

- 1) $QQ^\dagger Q = Q$
- 2) $Q^\dagger QQ^\dagger = Q^\dagger$
- 3) $(QQ^\dagger)' = QQ^\dagger$
- 4) $(Q^\dagger Q)' = Q^\dagger Q$.

Assumption 2.6. *Assume the Lipschitz constant l_2 of the driver f given in (9) satisfies*

$$l_2 \|\Psi_t^\dagger\|_{N \times N} \sqrt{6m} \leq 1, \quad \text{for any } t \in [0, T],$$

where Ψ is given in (6) and $m > 0$ is the bound of $\|A_t\|_{N \times N}$, for any $t \in [0, T]$.

The following lemma, which is a comparison result for BSDEs driven by a Markov chain, is found in Yang, Ramarimbahoaka and Elliott [17].

Lemma 2.7. *For $i = 1, 2$, suppose $(Y^{(i)}, Z^{(i)})$ is the solution of the BSDE:*

$$Y_t^{(i)} = \xi_i + \int_t^T f_i(s, Y_s^{(i)}, Z_s^{(i)}) ds - \int_t^T (Z_s^{(i)})' dM_s, \quad t \in [0, T].$$

Assume $\xi_1, \xi_2 \in L^2(\mathcal{F}_T)$, and $f_1, f_2 : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ satisfy some conditions such that the above two BSDEs have unique solutions. Moreover assume f_1 satisfies (9) and Assumption 2.6. If $\xi_1 \leq \xi_2$, a.s. and $f_1(t, Y_t^{(2)}, Z_t^{(2)}) \leq f_2(t, Y_t^{(2)}, Z_t^{(2)})$, a.e., a.s., then

$$P(Y_t^{(1)} \leq Y_t^{(2)}, \quad \text{for any } t \in [0, T]) = 1.$$

Lemma 2.8 is proved in Lepeltier and San Martin [11] (Lemma 1).

Lemma 2.8. *Assume $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function with linear growth, in the sense that there exists a constant $K \in (0, +\infty)$ such that for any $y \in \mathbb{R}$, $|f(y)| \leq K(1 + |y|)$. Then the sequence of functions*

$$f_n(y) = \inf_{u \in \mathbb{Q}} \{f(u) + n|y - u|\}$$

is well defined for $n \geq K$ and satisfies:

- (1) *linear growth: for any $y \in \mathbb{R}$, $|f_n(y)| \leq K(1 + |y|)$;*

- (2) *monotonicity in n* : for any $y \in \mathbb{R}$, $f_n(y) \nearrow$;
(3) *a Lipschitz continuous condition*: for any $y, u \in \mathbb{R}$,

$$|f_n(y) - f_n(u)| \leq n|y - u|;$$

- (4) *strong convergence*: if $y_n \rightarrow y$, $n \rightarrow +\infty$ then $f_n(y_n) \rightarrow f(y)$, $n \rightarrow +\infty$.

Lemma 2.9 can be found in Page 89 in Royden and Fitzpatrick [14] or in Page 172 in Yan and Liu [16].

Lemma 2.9. (*General Lebesgue Dominated Convergence Theorem*) Let $\{\eta_n\}_{n \in \mathbb{N}}$ and $\{\zeta_n\}_{n \in \mathbb{N}}$ be two sequences of random variables satisfying for any $n \in \mathbb{N}$, $|\eta_n| \leq \zeta_n$ and ζ_n is integrable. Suppose there exists an integrable random variable ζ such that $\zeta_n \rightarrow \zeta$, a.e., and $E[\zeta_n] \rightarrow E[\zeta]$. If $\eta_n \rightarrow \eta$, a.e., then

$$E[|\eta_n - \eta|] \rightarrow 0, \quad \text{moreover,} \quad E[\eta_n] \rightarrow E[\eta].$$

3 Existence Theorem of Multiple Solutions to BSDEs in Markov Chains with Continuous Coefficients and a Corresponding Comparison Result

Consider the following BSDE driven by a Markov chain

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z'_s dM_s, \quad t \in [0, T]. \quad (11)$$

Theorem 3.1. Assume $\xi \in L^2(\mathcal{F}_T)$ and $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ is a $\mathcal{P} \times \mathcal{B}(\mathbb{R}^{1+N})$ measurable function satisfying

- (i) *linear growth in y* : there exists a constant $K > 0$ such that for each $\omega \in \Omega, t \in [0, T], y \in \mathbb{R}, z \in \mathbb{R}^N$,

$$|f(w, t, y, z)| \leq K(1 + |y|);$$

- (ii) *Lipschitz in $z \in \mathbb{R}^N$* : there exists a constant $c_2 > 0$ such that, for any $t \in [0, T], y \in \mathbb{R}$ and $z, z' \in \mathbb{R}^N$:

$$|f(w, t, y, z) - f(w, t, y, z')| \leq c_2 \|z - z'\|_{X_t},$$

with c_2 satisfying

$$l_2 \|\Psi_t^\dagger\|_{N \times N} \sqrt{6m} \leq 1, \quad \text{for any } t \in [0, T],$$

where Ψ is given in (6) and $m > 0$ is the bound of $\|A_t\|_{N \times N}$, for any $t \in [0, T]$; (iii) for fixed $(\omega, t) \in \Omega \times [0, T]$, $f(\omega, t, \cdot, \cdot)$ is continuous.

Then there exists a solution $(Y, Z) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}) \times P^2_{\mathcal{F}}(0, T; \mathbb{R}^N)$ of BSDE (11).

We follow Lepeltier and San Martin [11], who discuss the case of a continuous BSDE driven by Brownian motion, and proceed with the proof of an existence result for equation (11).

Proof. Define for any $n \in \mathbb{N}$, $n \geq K$, $t \in [0, T]$ and $(y, z) \in \mathbb{R} \times \mathbb{R}^N$ the sequence:

$$f_n(t, y, z) = \inf_{u \in \mathbb{Q}} \{f(t, u, z) + n|y - u|\}.$$

From Lemma 2.8, we have for any $t \in [0, T]$, $y, y' \in \mathbb{R}$ and $z, z' \in \mathbb{R}^N$,

$$\begin{aligned} & \sup_{u \in \mathbb{Q}} \{f(t, u, z) - f(t, u, z')\} - \inf_{u \in \mathbb{Q}} \{f(t, u, z) + n|y' - u|\} \\ &= \sup_{u \in \mathbb{Q}} \{f(t, u, z) - f(t, u, z')\} + \sup_{u \in \mathbb{Q}} \{-f(t, u, z) - n|y' - u|\} \\ &\geq \sup_{u \in \mathbb{Q}} \{f(t, u, z) - f(t, u, z') - f(t, u, z) - n|y' - u|\} \\ &= \sup_{u \in \mathbb{Q}} \{-f(t, u, z') - n|y' - u|\} \\ &= - \inf_{u \in \mathbb{Q}} \{f(t, u, z') + n|y' - u|\}. \end{aligned}$$

Then

$$\begin{aligned} & f_n(t, y, z) - f_n(t, y', z') \\ &= f_n(t, y, z) - f_n(t, y', z) + f_n(t, y', z) - f_n(t, y', z') \\ &\leq n|y - y'| + \inf_{u \in \mathbb{Q}} \{f(t, u, z) + n|y' - u|\} - \inf_{u \in \mathbb{Q}} \{f(t, u, z') + n|y' - u|\} \\ &\leq n|y - y'| + \sup_{u \in \mathbb{Q}} \{f(t, u, z) - f(t, u, z')\} \\ &\leq n|y - y'| + \sup_{u \in \mathbb{Q}} \{c_2 \|z - z'\|_{X_t}\} \\ &= n|y - y'| + c_2 \|z - z'\|_{X_t}. \end{aligned}$$

Hence, interchanging the roles of (y, z) and (y', z') , we know for any $n \in \mathbb{N}$, $n \geq K$, $t \in [0, T]$, $y, y' \in \mathbb{R}$ and $z, z' \in \mathbb{R}^N$,

$$|f_n(t, y, z) - f_n(t, y', z')| \leq n|y - y'| + c_2 \|z - z'\|_{X_t}.$$

By Lemma 2.4 and Lemma 2.8, for any $n \in \mathbb{N}$, $n \geq K$, we deduce that the BSDE

$$Y_t^{(n)} = \xi + \int_t^T f_n(s, Y_s^{(n)}, Z_s^{(n)}) ds - \int_t^T (Z_s^{(n)})' dM_s, \quad t \in [0, T]$$

has a unique solution $(Y^{(n)}, Z^{(n)}) \in L_{\mathcal{F}}^2(0, T; \mathbb{R}) \times P_{\mathcal{F}}^2(0, T; \mathbb{R}^N)$. So for any $n \in \mathbb{N}$, $n \geq K$, we know $|Y_t^{(n)}| < +\infty$, a.e., a.s. For $t \in [0, T]$, $y \in \mathbb{R}$, $z \in \mathbb{R}^N$, define

$$\psi(t, y, z) = K(1 + |y|).$$

Then for any $t \in [0, T]$, $\psi(t, y, z)$ is a Lipschitz function in (y, z) and also by Lemma 2.4, we derive that the BSDE

$$U_t = \xi + \int_t^T \psi(s, U_s, V_s) ds - \int_t^T (V_s)' dM_s, \quad t \in [0, T]$$

has a unique solution $(U, V) \in L_{\mathcal{F}}^2(0, T; \mathbb{R}) \times P_{\mathcal{F}}^2(0, T; \mathbb{R}^N)$. Thus $|U_t| < +\infty$, a.e., a.s. By Lemma 2.8, we have $f_1 \leq f_2 \leq \dots \leq \psi$. Then by (ii) and Lemma 2.7, we have for any $n \in \mathbb{N}$, $n \geq K$, there exists a subset $A_n \subseteq \Omega$ with $P(A_n) = 1$ such that for any $\omega \in A_n$,

$$Y_t^{(n)}(\omega) \leq Y_t^{(n+1)}(\omega), \quad \text{for any } t \in [0, T].$$

Moreover, for any $n \in \mathbb{N}$, $n \geq K$, there exists a subset $B_n \subseteq \Omega$ with $P(B_n) = 1$ such that for any $\omega \in B_n$,

$$Y_t^{(n)}(\omega) \leq U_t(\omega), \quad \text{for any } t \in [0, T].$$

Hence

$$\begin{aligned} P\left(\bigcap_{n=K}^{+\infty} (A_n \cap B_n)\right) &= 1 - P\left(\bigcup_{n=K}^{+\infty} (A_n^c \cup B_n^c)\right) \\ &\geq 1 - \sum_{n=K}^{+\infty} (P(A_n^c) + P(B_n^c)) \\ &= 1. \end{aligned}$$

That is,

$$P(Y_t^{(K)} \leq Y_t^{(K+1)} \leq \dots \leq U_t, \quad \text{for any } t \in [0, T]) = 1.$$

For any $\omega \in \Omega$, $t \in [0, T]$, set

$$Y_t(\omega) = \sup_{n \in \mathbb{N}, n \geq K} Y_t^{(n)}(\omega).$$

Then

$$|Y_t| \leq |Y_t^{(K)}| + |U_t|, \text{ a.e., a.s.,}$$

hence, $E[\int_0^T |Y_t|^2 dt] < +\infty$. Moreover, $|Y_t| < +\infty$, a.e., a.s. Since

$$|Y_t^{(n)} - Y_t| \searrow 0, \text{ a.e., a.s.}$$

when $n \rightarrow +\infty$, by Levi's lemma we have

$$E[\int_0^T |Y_t^{(n)} - Y_t|^2 dt] \rightarrow 0, \quad n \rightarrow +\infty.$$

Lemma 3.2. *There exists a constant $C > 0$, such that*

$$\sup_{n \in \mathbb{N}, n \geq K} E[\int_0^T (|Y_t^{(n)}|^2 + \|Z_t^{(n)}\|_{X_t}^2) dt] \leq C. \quad (12)$$

Proof. By the Stieltjes Chain rule, we known for any $n \in \mathbb{N}$, $n \geq K$,

$$\begin{aligned} |Y_t^{(n)}|^2 &= |\xi|^2 + 2 \int_t^T Y_s^{(n)} f_n(s, Y_s^{(n)}, Z_s^{(n)}) ds \\ &\quad - 2 \int_t^T Y_{s-}^{(n)} (Z_s^{(n)})' dM_s - \sum_{t < s \leq T} \Delta Y_s^n \Delta Y_s^n. \end{aligned}$$

Because $\Delta M_s = \Delta X_s$, we have

$$\begin{aligned} \sum_{t < s \leq T} \Delta Y_s^{(n)} \Delta Y_s^{(n)} &= \sum_{t < s \leq T} ((Z_s^{(n)})' \Delta M_s) ((Z_s^{(n)})' \Delta M_s) \\ &= \sum_{t < s \leq T} (Z_s^{(n)})' \Delta X_s \Delta X_s' Z_s^{(n)} \\ &= \int_t^T (Z_s^{(n)})' (dL_s + d\langle X, X \rangle_s) Z_s^{(n)} \\ &= \int_t^T (Z_s^{(n)})' dL_s Z_s^{(n)} + \int_t^T \|Z_s^{(n)}\|_{X_s}^2 ds. \end{aligned}$$

Let $\beta > 0$ be an arbitrary constant. Using the product rule for $e^{\beta t} |Y_t^{(n)}|^2$ and from the above equation we derive for any $n \in \mathbb{N}$, $n \geq K$,

$$\begin{aligned} &E[|Y_0^{(n)}|^2] + E[\int_0^T \beta |Y_s^{(n)}|^2 e^{\beta s} ds] + E[\int_0^T e^{\beta s} \|Z_s^{(n)}\|_{X_s}^2 ds] \\ &= E[e^{\beta T} |\xi|^2] + 2E[\int_0^T e^{\beta s} Y_s^{(n)} f_n(s, Y_s^{(n)}, Z_s^{(n)}) ds] \\ &\leq E[e^{\beta T} |\xi|^2] + 2E[\int_0^T e^{\beta s} |Y_s^{(n)}| (K(1 + |Y_s^{(n)}|)) ds] \end{aligned}$$

$$\begin{aligned}
&\leq E[e^{\beta T}|\xi|^2] + 2E\left[\int_0^T e^{\beta s} K|Y_s^{(n)}|ds\right] + 2KE\left[\int_0^T e^{\beta s}|Y_s^{(n)}|^2ds\right] \\
&\leq E[e^{\beta T}|\xi|^2] + K^2Te^{\beta T} + E\left[\int_0^T e^{\beta s}|Y_s^{(n)}|^2ds\right] + 2KE\left[\int_0^T e^{\beta s}|Y_s^{(n)}|^2ds\right] \\
&\leq E[e^{\beta T}|\xi|^2] + K^2Te^{\beta T} + (1+2K)E\left[\int_0^T e^{\beta s}|Y_s^{(n)}|^2ds\right]
\end{aligned}$$

Set $\beta = 2K + 2$. Then, we obtain for any $n \in \mathbb{N}$, $n \geq K$,

$$E\left[\int_0^T |Y_s^{(n)}|^2 e^{\beta s} ds\right] + E\left[\int_0^T e^{\beta s} \|Z_s^{(n)}\|_{X_s}^2 ds\right] \leq E[e^{\beta T}|\xi|^2] + K^2Te^{\beta T}.$$

So

$$\begin{aligned}
&E\left[\int_0^T |Y_s^{(n)}|^2 ds\right] + E\left[\int_0^T \|Z_s^{(n)}\|_{X_s}^2 ds\right] \\
&\leq E\left[\int_0^T |Y_s^{(n)}|^2 e^{\beta s} ds\right] + E\left[\int_0^T e^{\beta s} \|Z_s^{(n)}\|_{X_s}^2 ds\right] \\
&\leq E[e^{\beta T}|\xi|^2] + K^2Te^{\beta T}.
\end{aligned}$$

Since the last line of the above inequality does not depend on n , we conclude that there exists a constant $C > 0$ such that for all $n \in \mathbb{N}$, $n \geq K$, inequality (12) holds. \square

We continue the proof of Theorem 3.1. For any $n, p \in \mathbb{N}$, $n, p \geq K$, using the product rule for $|Y_t^{(n)} - Y_t^{(p)}|^2$,

$$\begin{aligned}
&|Y_t^{(n)} - Y_t^{(p)}|^2 \\
&= 2 \int_t^T (Y_s^{(n)} - Y_s^{(p)})(f_n(s, Y_s^{(n)}, Z_s^{(n)}) - f_p(s, Y_s^{(p)}, Z_s^{(p)}))ds \\
&\quad - 2 \int_t^T (Y_{s-}^{(n)} - Y_{s-}^{(p)})(Z_s^{(n)} - Z_s^{(p)})' dM_s - \sum_{t < s \leq T} \Delta(Y_s^{(n)} - Y_s^{(p)})\Delta(Y_s^{(n)} - Y_s^{(p)}).
\end{aligned}$$

Here,

$$\begin{aligned}
&\sum_{t < s \leq T} \Delta(Y_s^{(n)} - Y_s^{(p)})\Delta(Y_s^{(n)} - Y_s^{(p)}) \\
&= \sum_{t < s \leq T} ((Z_s^{(n)} - Z_s^{(p)})' \Delta M_s)((Z_s^{(n)} - Z_s^{(p)})' \Delta M_s) \\
&= \sum_{t < s \leq T} (Z_s^{(n)} - Z_s^{(p)})' \Delta X_s \Delta X_s' (Z_s^{(n)} - Z_s^{(p)})
\end{aligned}$$

$$\begin{aligned}
&= \int_t^T (Z_s^{(n)} - Z_s^{(p)})'(dL_s + d\langle X, X \rangle_s)(Z_s^{(n)} - Z_s^{(p)}) \\
&= \int_t^T (Z_s^{(n)} - Z_s^{(p)})'dL_s(Z_s^{(n)} - Z_s^{(p)}) + \int_t^T \|Z_s^{(n)} - Z_s^{(p)}\|_{X_s}^2 ds.
\end{aligned}$$

Set $t = 0$, taking the expectation on both sides of the above equation, we deduce

$$\begin{aligned}
&E[|Y_0^{(n)} - Y_0^{(p)}|^2] + E\left[\int_0^T \|Z_s^{(n)} - Z_s^{(p)}\|_{X_s}^2 ds\right] \\
&= 2E\left[\int_0^T (Y_s^{(n)} - Y_s^{(p)})(f_n(s, Y_s^{(n)}, Z_s^{(n)}) - f_p(s, Y_s^{(p)}, Z_s^{(p)}))ds\right].
\end{aligned}$$

So by Lemma 3.2 we know there exists a constant $C' > 0$ depending on the constant C given in Lemma 3.2 such that

$$\begin{aligned}
&E\left[\int_0^T \|Z_s^{(n)} - Z_s^{(p)}\|_{X_s}^2 ds\right] \\
&\leq 2E\left[\int_0^T (Y_s^{(n)} - Y_s^{(p)})(f_n(s, Y_s^{(n)}, Z_s^{(n)}) - f_p(s, Y_s^{(p)}, Z_s^{(p)}))ds\right] \\
&\leq 2(E\left[\int_0^T |Y_s^{(n)} - Y_s^{(p)}|^2 ds\right])^{\frac{1}{2}}(E\left[\int_0^T |f_n(s, Y_s^{(n)}, Z_s^{(n)}) - f_p(s, Y_s^{(p)}, Z_s^{(p)})|^2 ds\right])^{\frac{1}{2}} \\
&\leq 2(E\left[\int_0^T |Y_s^{(n)} - Y_s^{(p)}|^2 ds\right])^{\frac{1}{2}}(E\left[\int_0^T (|f_n(s, Y_s^{(n)}, Z_s^{(n)})| + |f_p(s, Y_s^{(p)}, Z_s^{(p)})|)^2 ds\right])^{\frac{1}{2}} \\
&\leq 2(E\left[\int_0^T |Y_s^{(n)} - Y_s^{(p)}|^2 ds\right])^{\frac{1}{2}}K(E\left[\int_0^T (2 + |Y_s^{(n)}| + |Y_s^{(p)}|)^2 ds\right])^{\frac{1}{2}} \\
&\leq 2(E\left[\int_0^T |Y_s^{(n)} - Y_s^{(p)}|^2 ds\right])^{\frac{1}{2}}K(3E\left[\int_0^T (4 + |Y_s^{(n)}|^2 + |Y_s^{(p)}|^2) ds\right])^{\frac{1}{2}} \\
&\leq 2KC'(E\left[\int_0^T |Y_s^{(n)} - Y_s^{(p)}|^2 ds\right])^{\frac{1}{2}}.
\end{aligned}$$

Hence, $\{Z^{(n)}, n \in \mathbb{N}, n \geq K\}$ is a Cauchy sequence in $P_{\mathcal{F}}^2(0, T; \mathbb{R}^N)$. Consider the factor space of equivalence classes of processes in $P_{\mathcal{F}}^2(0, T; \mathbb{R}^N)$. An equivalence class is just all processes which differ by a null process. On that space the semi norm is actually a norm and so the space is complete. Then there exists a process $Z \in P_{\mathcal{F}}^2(0, T; \mathbb{R}^N)$ such that

$$E\left[\int_0^T \|Z_t^{(n)} - Z_t\|_{X_t}^2 dt\right] \rightarrow 0, \quad n \rightarrow +\infty. \quad (13)$$

Also,

$$\begin{aligned}
& |f_n(t, Y_t^n, Z_t^n) - f(t, Y_t, Z_t)| \\
& \leq |f_n(t, Y_t^n, Z_t^n) - f_n(t, Y_t^n, Z_t)| + |f_n(t, Y_t^n, Z_t) - f(t, Y_t, Z_t)| \\
& \leq c_2 \|Z_t^n - Z_t\|_{X_t} + |f_n(t, Y_t^n, Z_t) - f(t, Y_t, Z_t)|.
\end{aligned}$$

Thus by (13) and Lemma 2.8 (4), we have

$$f_n(t, Y_t^{(n)}, Z_t^{(n)}) \rightarrow f(t, Y_t, Z_t), \quad a.e., a.s.$$

Let Q be a probability on $\Omega \times [0, T]$ satisfying $Q|_{\Omega} = P$ and $dQ|_{[0, T]} = \frac{dt}{T}$. Denote the expectation under Q by $E^Q[\cdot]$. Thus for any $n \in \mathbb{N}$, $n \geq K$, we obtain

$$E^Q[K(1 + |Y_t^{(n)}|)] < +\infty.$$

Moreover, when $n \rightarrow +\infty$, we derive

$$\begin{aligned}
(1) \quad & f_n(t, Y_t^{(n)}, Z_t^{(n)}) \rightarrow f(t, Y_t, Z_t), \quad Q\text{-a.e.}; \\
(2) \quad & K(1 + |Y_t^{(n)}|) \rightarrow K(1 + |Y_t|), \quad Q\text{-a.e.}; \\
(3) \quad & E^Q[K(1 + |Y_t^{(n)}|)] \rightarrow E^Q[K(1 + |Y_t|)] < +\infty.
\end{aligned}$$

Since for any $n \in \mathbb{N}$, $n \geq K$, $t \in [0, T]$,

$$f_n(t, Y_t^{(n)}, Z_t^{(n)}) \leq K(1 + |Y_t^{(n)}|),$$

we have by Lemma 2.9, when $n \rightarrow +\infty$,

$$E^Q[|f_n(t, Y_t^{(n)}, Z_t^{(n)}) - f(t, Y_t, Z_t)|] \rightarrow 0,$$

hence,

$$E\left[\int_0^T |f_n(t, Y_t^{(n)}, Z_t^{(n)}) - f(t, Y_t, Z_t)| dt\right] \rightarrow 0.$$

Therefore, we conclude when $n \rightarrow +\infty$,

$$\begin{aligned}
& E\left[\sup_{t \in [0, T]} \left| \int_t^T f_n(s, Y_s^{(n)}, Z_s^{(n)}) ds - \int_t^T f(s, Y_s, Z_s) ds \right|\right] \\
& \leq E\left[\sup_{t \in [0, T]} \int_t^T |f_n(s, Y_s^{(n)}, Z_s^{(n)}) - f(s, Y_s, Z_s)| ds\right] \\
& = E\left[\int_0^T |f_n(s, Y_s^{(n)}, Z_s^{(n)}) - f(s, Y_s, Z_s)| ds\right] \rightarrow 0.
\end{aligned}$$

By Doob's martingale inequality and Lemma 2.3, we know when $n \rightarrow +\infty$,

$$\begin{aligned}
& E\left[\sup_{t \in [0, T]} \left| \int_t^T (Z_s^{(n)} - Z_s)' dM_s \right|^2\right] \\
&= E\left[\sup_{t \in [0, T]} \left| \int_0^T (Z_s^{(n)} - Z_s)' dM_s - \int_0^t (Z_s^{(n)} - Z_s)' dM_s \right|^2\right] \\
&\leq 2E\left[\left| \int_0^T (Z_s^{(n)} - Z_s)' dM_s \right|^2\right] + \sup_{t \in [0, T]} \left| \int_0^t (Z_s^{(n)} - Z_s)' dM_s \right|^2 \\
&\leq 10E\left[\left| \int_0^T (Z_s^{(n)} - Z_s)' dM_s \right|^2\right] = 10E\left[\int_0^T \|Z_s^{(n)} - Z_s\|_{X_s}^2 ds\right] \rightarrow 0.
\end{aligned}$$

So (Y, Z) satisfies BSDE (11). \square

Theorem 3.3. *We make the same assumptions as in Theorem 3.1. Then there is a minimal solution \bar{Y} of (11), in the sense that for any other solution Y of (11), we have*

$$P(\bar{Y}_t \leq Y_t, \text{ for any } t \in [0, T]) = 1.$$

Proof. By Theorem 3.1, there is a solution $(Y', Z') \in L_{\mathcal{F}}^2(0, T; \mathbb{R}) \times P_{\mathcal{F}}^2(0, T; \mathbb{R}^N)$ of BSDE (11). By Lemma 2.7, we have for any $n \in \mathbb{N}, n \geq K$,

$$P(Y_t^{(n)} \leq Y'_t, \text{ for any } t \in [0, T]) = 1,$$

here $Y_t^{(n)}$ is the same as in the proof of Theorem 3.1. That is, for any $n \in \mathbb{N}, n \geq K$, there is a subset $F_n \subseteq \Omega$ such that $P(F_n) = 1$ and for all $\omega \in F_n$, $Y_t^{(n)} \leq Y'_t$, for any $t \in [0, T]$. Thus,

$$\begin{aligned}
& P(Y_t \leq Y'_t, \text{ for any } t \in [0, T]) \\
&= P\left(\sup_{n \in \mathbb{N}, n \geq K} Y_t^{(n)} \leq Y'_t, \text{ for any } t \in [0, T]\right) \\
&= P\left(\bigcap_{n=K}^{+\infty} F_n\right) = 1 - P\left(\bigcup_{n=K}^{+\infty} F_n^c\right) \\
&\geq 1 - \sum_{n=K}^{+\infty} P(F_n^c) = 1,
\end{aligned}$$

that is, Y is the minimal solution. \square

Since the solutions of BSDEs with Markov chain noise and continuous coefficients are not unique, we cannot give comparison results for all solutions. However, noticing the minimal solution is unique for a BSDE of this kind we can compare the minimal solutions of these BSDEs.

Consider the following two BSDEs for Markov chain noise:

$$Y_t = \xi_1 + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z'_s dM_s, \quad t \in [0, T]$$

and

$$U_t = \xi_2 + \int_t^T g(s, U_s, V_s) ds - \int_t^T V_s' dM_s, \quad t \in [0, T].$$

Theorem 3.4. *Assume $\xi_1, \xi_2 \in L^2(\mathcal{F}_T)$, f and g both satisfy all the conditions of Theorem 3.1. Denote the minimal solutions of the above two BSDEs by \bar{Y} and \bar{U} , respectively. If $\xi_1 \leq \xi_2$, and for any $t \in [0, T], y \in \mathbb{R}, z \in \mathbb{R}^N$, $f(t, y, z) \leq g(t, y, z)$, then*

$$P(\bar{Y}_t \leq \bar{U}_t, \text{ for all } t \in [0, T]) = 1.$$

Proof. Similarly to the proof of Theorem 3.1, there exists a constant $K' \in (0, +\infty)$ such that we can denote for fixed $(t, \omega) \in [0, T] \times \Omega$, the sequence associated with $f(t, y, z)$ by $f_n(t, y, z)$, $n \in \mathbb{N}, n \geq K'$, the sequence associated with $g(t, y, z)$ by $g_n(t, y, z)$, $n \in \mathbb{N}, n \geq K'$. Then we have for any $n \in \mathbb{N}, n \geq K', t \in [0, T], y \in \mathbb{R}, z \in \mathbb{R}^N$, $f_n(t, y, z) \leq g_n(t, y, z)$. By Lemma 2.4, for any $n \in \mathbb{N}, n \geq K'$, we deduce that the BSDE

$$Y_t^{(n)} = \xi_1 + \int_t^T f_n(s, Y_s^{(n)}, Z_s^{(n)}) ds - \int_t^T (Z_s^{(n)})' dM_s, \quad t \in [0, T]$$

has a unique solution $(Y^{(n)}, Z^{(n)}) \in L_{\mathcal{F}}^2(0, T; \mathbb{R}) \times P_{\mathcal{F}}^2(0, T; \mathbb{R}^N)$ and the BSDE

$$U_t^{(n)} = \xi_2 + \int_t^T g_n(s, U_s^{(n)}, V_s^{(n)}) ds - \int_t^T (V_s^{(n)})' dM_s, \quad t \in [0, T]$$

has a unique solution $(U^{(n)}, V^{(n)}) \in L_{\mathcal{F}}^2(0, T; \mathbb{R}) \times P_{\mathcal{F}}^2(0, T; \mathbb{R}^N)$. By Lemma 2.7, we obtain for any $n \in \mathbb{N}, n \geq K'$, there exists there exists a subset $A_n \subseteq \Omega$ with $P(A_n) = 1$ such that for any $\omega \in A_n$,

$$Y_t^{(n)}(\omega) \leq U_t^{(n)}(\omega), \quad \text{for all } t \in [0, T].$$

Similarly to the proof of Theorem 3.1 we know $P(\bigcap_{n=K}^{+\infty} A_n) = 1$. That is,

$$P(Y_t^{(n)} \leq U_t^{(n)}, \text{ for all } n \in \mathbb{N}, n \geq K', t \in [0, T]) = 1.$$

So for a.e. $\omega \in \Omega$,

$$\bar{Y}_t = \sup_{n \in \mathbb{N}, n \geq K'} Y_t^{(n)} \leq \sup_{n \in \mathbb{N}, n \geq K'} U_t^{(n)} = \bar{U}_t, \text{ for all } t \in [0, T].$$

□

4 Application to European Options

It is shown in [6], for a market where the underlying securities follow a geometric Brownian motion model, that the pricing of a European option can be formulated in terms of BSDEs driven by a Brownian motion.

In this section, T will be the time horizon. We consider a market composed of a bond S^0 , whose price dynamics are

$$dS_t^0 = r_t S_t^0 dt, \quad t \in [0, T],$$

and N stocks S^i , $i = 1, \dots, N$ whose price dynamics are

$$dS_t^i = S_{t-}^i (g_t^i dt + \sum_{j=1}^N h_t^{ij} dM_t^j), \quad t \in [0, T].$$

Here, at any time $t \in [0, T]$, r_t is the interest rate, $g_t^i \in \mathbb{R}$ is the appreciation rate of the stock S^i and $h_t = (h_t^{ij}) \in \mathbb{R}^{N \times N}$ is the volatility matrix. We assume that:

1. The interest rate r is a non-negative predictable process.
2. The appreciation rate g is a predictable process in \mathbb{R}^N .
3. The volatility h is also a predictable process in $\mathbb{R}^{N \times N}$ and is invertible.

For $i = 1, \dots, N$, write $\bar{S}_t^i = e^{-\int_0^t r_s ds} S_t^i$ for the discounted stock price at time $t \in [0, T]$.

No-arbitrage assumption

Recall from asset pricing theory that the existence of an equivalent martingale measure ensures no-arbitrage. That is, we need to find a measure Q equivalent to P under which, for each i , the discounted price \bar{S}^i is a martingale.

The following lemmas are from [7].

Lemma 4.1. *Suppose $\{Y_t\}_{t \geq 0}$, is a semimartingale and suppose $X_{0-} = 0$ a.s. Then there is a unique semimartingale $\{Z_t\}$ such that*

$$Z_t = Z_{0-} + \int_0^t Z_{s-} dX_s.$$

Furthermore, Z_t is given by the expression

$$Z_t = Z_{0-} \exp(X_t - \frac{1}{2} \langle X^c, X^c \rangle_t) \prod_{0 \leq s \leq t} (1 + \Delta X_s) e^{-\Delta X_s},$$

for $t \geq 0$, where the infinite product is absolutely convergent almost surely.

Lemma 4.2. (*Girsanov Transformation*) Let Q be an equivalent measure to P and put N_t as the RCLL version of $\{E[\frac{dQ}{dP}|\mathcal{F}_t], t \geq 0\}$. If \hat{M} is a P -local martingale such that $\hat{M}_0 = 0$ then $-\int_0^t \frac{1}{N} d[N, \hat{M}] + \hat{M}$ is a local martingale under the measure Q .

In our discussion, we assume that there exists a predictable process $\theta_t \in \mathbb{R}^N$ such that:

$$g_t - r_t \mathbf{1} = h_t \theta_t, \quad (14)$$

with $|\theta| \leq K_0$ for some constant $K_0 > 0$. For each $i = 1, \dots, N$, applying Itô's product rule to $e^{-\int_0^t r_s ds} S_t^i$, noting $[e^{-\int_0^t r_s ds}, S^i] = 0$ and using (14), we obtain for $i = 1, \dots, N$,

$$\begin{aligned} d\bar{S}_t^i &= de^{-\int_0^t r_s ds} S_t^i = -r_t e^{-\int_0^t r_s ds} S_t^i dt + e^{-\int_0^t r_s ds} dS_t^i \\ &= -r_t e^{-\int_0^t r_s ds} S_t^i dt + e^{-\int_0^t r_s ds} S_t^i g_t^i dt + e^{-\int_0^t r_s ds} S_{t-} \sum_{j=1}^N h_t^{ij} dM_t^j \\ &= \bar{S}_{t-}^i ((-r_t + g_t^i) dt + \sum_{j=1}^N h_t^{ij} dM_t^j) \\ &= \bar{S}_{t-}^i ((h_t \theta_t)^i dt + \sum_{j=1}^N h_t^{ij} dM_t^j) \\ &= \bar{S}_{t-}^i (\sum_{j=1}^N h_t^{ij} (\theta_t^j dt + dM_t^j)). \end{aligned}$$

Let Y_t be the vector process satisfying

$$dY_t = h_t (\theta_t dt + dM_t). \quad (15)$$

Note, Y is a semimartingale and from Lemma 4.1, the unique solution to

$$d\bar{S}_t^i = \bar{S}_{t-}^i dY_t^i, \quad i = 1, \dots, N,$$

is the stochastic exponential $\bar{S}_t^i = \bar{S}_0^i \mathcal{E}(Y^i)_t$, where

$$\mathcal{E}(Y^i)_t = \exp(Y_t^i - \frac{1}{2} \langle (Y^i)^c, (Y^i)^c \rangle_t) \prod_{0 \leq s \leq t} (1 + \Delta Y_s^i) e^{-\Delta Y_s^i}.$$

Lemma 4.3. If Y given in (15) is a martingale under some measure Q equivalent to P then $\mathcal{E}(Y)$ is also a martingale under Q .

Proof. The proof is straightforward since for $i = 1, \dots, N$, $\bar{S}_t^i = \bar{S}_0^i + \int_0^t \bar{S}_s^i dY_s^i$ and $\int_0^t \bar{S}_s^i dY_s^i$ is a martingale, hence \bar{S}_t^i is a martingale, so is \bar{S}_t therefore $\mathcal{E}(Y)$ is also a martingale. \square

Now, write $\hat{M}_t = \int_0^t h_s dM_s$. Then \hat{M} is a martingale with $\hat{M}_0 = 0$. Suppose there is a uniformly integrable martingale process N satisfying

$$-\frac{1}{N_t} d[N, \hat{M}]_t = h_t \theta_t dt, \quad \text{for any } t \in [0, T],$$

and an equivalent measure Q such that $N_t = E \left[\frac{dQ}{dP} | \mathcal{F}_t \right]$, then by Lemma 4.2, Y in (15) is a martingale under Q and so is \bar{S} , by Lemma 4.3. Q is then an equivalent martingale measure for the market which ensures there is no-arbitrage opportunity.

Now, assume that investors consume continuously a part of their wealth or profit. A consumption rate, at time t , for an investor is denoted by c_t which is an adapted process and the cumulative spending is $C_t = \int_0^t c_s ds$. In our discussion, we consider investors who decide to limit their consumption, that is a positive constant amount K_1 is chosen by each investor such that $|c_t| < K_1$.

Given the consumption rate model, we shall find a strategy which replicates the European option at the exercise time $T < \infty$. In other words, we shall determine the amount of money that we shall invest in the above securities in order to be able to pay off the option at maturity time T . Therefore, here, a strategy is a couple $(V, (\pi^0, \pi))$ where $V \in \mathbb{R}$ is the portfolio value, π_t^0 is number of bonds held at time t and $\pi_t = (\pi_t^1, \dots, \pi_t^N)$ such that for $i = 1, \dots, N$, π_t^i is the number of stocks i held at time t . The portfolio value V of the investor at any time $t \in [0, T]$ is then $V_t = \sum_{i=0}^N \pi_t^i S_t^i$. The strategy needs to be self-financing, that is any increase and decrease in his wealth V comes from gains and losses from the investment and the consumption. Such a strategy satisfies

$$dV_t = \pi_t^0 dS_t^0 + \sum_{i=1}^N \pi_{t-}^i dS_t^i - dC_t.$$

Using the dynamics of S^0 , S^i , $i = 1, \dots, N$ and (14), we have:

$$\begin{aligned} dV_t &= \pi_t^0 r_t S_t^0 dt + \sum_{i=1}^N S_t^i \pi_t^i g_t^i dt + \sum_{i=1}^N \pi_{t-}^i S_{t-}^i \sum_{j=1}^N h_t^{ij} dM_t^j - dC_t \\ &= r_t (V_t - \sum_{i=1}^N \pi_t^i S_t^i) dt + \sum_{i=1}^N S_t^i \pi_t^i g_t^i dt + \sum_{i=1}^N \pi_t^i S_{t-}^i \sum_{j=1}^N h_t^{ij} dM_t^j - dC_t \end{aligned}$$

$$\begin{aligned}
&= r_t V_t dt + \sum_{i=1}^N \pi_t^i S_t^i (-r_t + g_t^i) dt + \sum_{i=1}^N \pi_{t-}^i S_{t-}^i \sum_{j=1}^N h_t^{ij} dM_t^j - dC_t \\
&= r_t V_t dt + \sum_{i=1}^N \pi_t^i S_t^i \sum_{j=1}^N h_t^{ij} \theta_t^j dt + \sum_{i=1}^N \pi_{t-}^i S_{t-}^i \sum_{j=1}^N h_t^{ij} dM_t^j - dC_t.
\end{aligned}$$

That is,

$$dV_t = r_t V_t + (\text{diag}(S_t) \pi_t)' h_t \theta_t dt + (\text{diag}(S_t) \pi_t)' h_t dM_t - dC_t.$$

Writing the backward integral form of the above, for $t \in [0, T]$, we have:

$$V_t = V_T + \int_t^T (c_s - r_s V_s - (\text{diag}(S_s) \pi_s)' h_s \theta_s) ds - \int_t^T (\text{diag}(S_{s-}) \pi_{s-})' h_s dM_s. \quad (16)$$

Now, we have the following definitions:

Definition 4.4. A hedging strategy for a European option whose payoff at time T is ξ , is a self-financing strategy (V, π) such that $V_T = \xi$ with $E[\int_0^T |h'_s \pi_s|_N^2 ds] < \infty$. If such a strategy exists, the European option is called hedgeable.

Denote the set of strategies given in the above definition by $\mathcal{S}(\xi)$ and the fair price at time t of the European option by P_t . We use the following definition from [9].

Definition 4.5. The fair price P_t at time t of the hedgeable option is the smallest amount needed to hedge the option. That is

$$P_t = \inf\{x \geq 0; \text{ there exists } (V, \pi) \in \mathcal{S}(\xi) \text{ such that } V_t = x\}.$$

Let

$$f(t, v, z) = c_t - r_t v - z' \theta_t. \quad (17)$$

For some constant $K_2 > 0$, we assume $|r_t| \leq K_2$. Take

$$K_3 = \max\{K_1, K_2, K_0 \sqrt{3m}\}.$$

From Lemma 2.2, for any $t \in [0, T]$

$$\begin{aligned}
|f(t, v, z)| &\leq |c_t| + |r_t| |v| + |\theta_t' z| \\
&\leq K_1 + K_2 |v| + K_0 \sqrt{3m} \|z\|_{X_t} \\
&\leq K_3 (1 + |v| + \|z\|_{X_t}).
\end{aligned}$$

Since investors can only hold a finite number of shares, it is reasonable to suppose that z is bounded. Therefore, there is a constant K'_3 such that

$$|f(t, v, z)| \leq K'_3(1 + |v|).$$

Hence f is linear increasing in y with constant K'_3 . Also f is Lipschitz in z with constant K_3 . Consequently, we have the following proposition:

Proposition 4.6. *Assume f in equation (17) satisfies $K'_3\|\Psi_t^\dagger\|_{N \times N}\sqrt{6m} \leq 1$. Let $\xi \in L^2(\mathcal{F}_T)$. Then $\mathcal{S}(\xi)$ is non-empty.*

Proof. By Theorem 3.1, a solution (V, Z) to the BSDE

$$V_t = \xi + \int_t^T (c_s - r_s V_s - Z'_s \theta_s) ds - \int_t^T Z'_s dM_s \quad (18)$$

exists. Let (V, Z) be such a solution, then a strategy satisfying (16) exists if there is a solution π_t to the equation $Z_t = h'_t \text{diag}(S_t) \pi_t$, for any $t \in [0, T]$. Since h and $\text{diag}(S_t)$ are invertible, the equation admits a unique solution π_t . Therefore, the set $\mathcal{S}(\xi)$ of hedging strategies is non-empty. \square

Proposition 4.7. *Assume f in equation (17) satisfies $K'_3\|\Psi_t^\dagger\|_{N \times N}\sqrt{6m} \leq 1$. The fair price P_t , at time $t \in [0, T]$ of the European option exists and satisfies:*

$$P_t = E \left[\xi + \int_t^T (c_s - r_s V_s - Z'_s \theta_s) ds \middle| \mathcal{F}_t \right],$$

where (V, Z) is one pair solution of (18). Moreover, V is minimal.

Proof. Theorem 3.3 ensures there is a minimal solution of (18). Let V be such a minimal solution. Then from Definition 4.5, $P_t = V_t$ is the fair price of the European option at any time $t \in [0, T]$. Taking the expectation in (18), given the information at time t , we obtain the result. \square

Proposition 4.8. *Let $(V^{(1)}, Z^{(1)})$ and $(V^{(2)}, Z^{(2)})$, be two solutions of (18). Write $\bar{V}_t^{(i)} = e^{-\int_0^t r_s ds} V_t^{(i)}$, $i = 1, 2$. Then for $t \in [0, T]$,*

$$E^Q \left[\bar{V}_t^{(1)} - \bar{V}_t^{(2)} \right] = 0$$

and

$$E^Q[\bar{V}_t^{(1)}] = E^Q[\bar{V}_t^{(2)}] = E^Q \left[\xi + \int_t^T e^{-\int_0^s r_u du} c_s ds \right].$$

Proof. For $i = 1, 2$, let $\pi^{(i)}$ be the solution of $Z_t^{(i)} = h'_t \text{diag}(S_t) \pi_t^{(i)}$, for any $t \in [0, T]$. Using the product rule on $e^{-\int_0^t r_s ds} V_t^{(i)}$ and using Y in (15), we have

$$\begin{aligned} d\bar{V}_t^{(i)} &= -e^{-\int_0^t r_s ds} c_t dt + e^{-\int_0^t r_s ds} (Z_t^{(i)})' (\theta_t dt + dM_t) \\ &= -e^{-\int_0^t r_s ds} c_t dt + e^{-\int_0^t r_s ds} (\text{diag}(S_t) \pi_t^{(i)})' h_t (\theta_t dt + dM_t) \\ &= -e^{-\int_0^t r_s ds} c_t dt + e^{-\int_0^t r_s ds} (\text{diag}(S_t) \pi_t^{(i)})' dY_t. \end{aligned}$$

Since Y is martingale under Q , for $t \in [0, T]$, integrating the above from t to T and taking the expectation, for $i = 1, 2$, we derive

$$E^Q[\bar{V}_t^{(i)}] = E^Q \left[\xi + \int_t^T e^{-\int_0^s r_u du} c_s ds \right].$$

This proves that under the measure Q , for all solutions (V, Z) of (18), all the V 's are equal, hence the price is unique under the risk neutral measure Q . \square

Conclusion

The paper discusses backward stochastic differential equations with Markov chain noise. Existence is established when the driver is not Lipschitz. The minimal solution is shown to be unique. The result is applied to pricing European options in a Markov chain market.

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